# Peak Order (Exercise 3)

$P_1 = 400 - q_1$ Peak Period Demand

$P_2 = 380 - q_2$ Off peak Demand

$b = 20$ b is the variable cost that the monopolist needs to pay in each market. b is a unitary cost.

$c = 10$ is the capacity cost.
c is paid only once and it's used in both markets.

Write down the Lagrangian and Kuhn Tucker conditions for this problem.
Find the optimal capacity and outputs for this problem.

How much of the capacity is paid for by each (Which are the values of $\lambda_1$ and $\lambda_2$)?

Now suppose capacity is $c = 30$ per unit.

Find quantities, capacity and how much of the capacity is paid for by each market. ($\lambda_1$ and $\lambda_2$)

$$\max_{q_1, q_2, K} (400-q_1)\cdot q_1 + (380-q_2)\cdot q_2 - 20(q_1+q_2) - 10K$$

s.t. $K \geq q_1$

$K \geq q_2$

Lagrangian is $\mathcal{L}(q_1, q_2, K, \lambda_1, \lambda_2) =$

$$(400-q_1)\cdot q_1 + (380-q_2)\cdot q_2 - 20(q_1+q_2) - 10K + \lambda_1(K-q_1) +$$

$$+ \lambda_2(K-q_2)$$

Kuhn Tucker conditions are

$$\frac{\partial \mathcal{L}}{\partial q_1} = 400 - 2q_1 - 20q_1 - \lambda_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = 380 - 2q_2 - 20q_2 - \lambda_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial K} = -10 + \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = K - q_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = K - q_2 = 0$$

Remember: $\lambda_1 \geq 0$, $\lambda_2 \geq 0$

Assume $q_1 > 0$, $q_2 > 0$, $K > 0$

Note that $P_1(q_1) > P_2(q_2)$, so since demand $P_2$

is lower than $P_1$ → Let's assume that $\lambda_2 = 0$

Intuitevely think as if you have a capacity problem.

it's going to be easier to reach that limit when you have a higher demand compared to a lower one.

Remember $\lambda=0$ means you are in the case of unconstrained maximization.

So given $\lambda_2=0$ we have:

$\frac{\partial \mathcal{L}}{\partial q_1} = 400 - 2q_1 - 20q_1 - \lambda_1 = 0 \rightarrow \frac{400-10}{22} = \frac{22}{22}q_1 \cong 17,72$

$\frac{\partial \mathcal{L}}{\partial q_2} = 380 - 2q_2 - 20q_2 - \lambda_2 = 0 \rightarrow \frac{380}{22} = \frac{\cancel{22}}{\cancel{22}}q_2 \cong 17,27$

$\frac{\partial \mathcal{L}}{\partial K} = -10 + \lambda_1 + \lambda_2 = 0 \rightarrow \lambda_1 = 10$

$\frac{\partial \mathcal{L}}{\partial \lambda_1} = K - q_1 = 0 \qquad K = q_1 = 17,72.$

$\frac{\partial \mathcal{L}}{\partial \lambda_2} = K - q_2 = 0$

So if $\lambda_2=0 \rightarrow$ from (3) implies that $\lambda_1>0 \rightarrow$ given $\lambda_2=0$ and $\lambda_1>0$ we obtained

$q_1^* = 17,27$ and $q_2^* = 17,72$ and $K^* = 17,72, \lambda_1^*=10, \lambda_2^*=0$

Does this values satisfy the constraints (4), (5)?

Yes! (4) is binding $\rightarrow$ $17,72 - 17,72 = 0$

(5) $17,72 - 17,27 > 0$ not binding!

so our assumptions about $\lambda_2=0$ and $\lambda_1>0$ are satisfied!

Now let's consider $\lambda_2 > 0, \lambda_1 > 0$ (4) (5) are binding

$\frac{\partial \mathcal{L}}{\partial q_1} = 400 - 2q_1 - 20q_1 - \lambda_1 = 0 \rightarrow 400 - 22q_1 - \lambda_1 = 0$

$\frac{\partial \mathcal{L}}{\partial q_2} = 380 - 2q_2 - 20q_2 - \lambda_2 = 0 \rightarrow 380 - 22q_1 - (10 - \lambda_1) = 0$

$\frac{\partial \mathcal{L}}{\partial K} = -10 + \lambda_1 + \lambda_2 = 0 \rightarrow \lambda_2 = 10 - \lambda_1$

$\left.\begin{array}{l} \frac{\partial \mathcal{L}}{\partial \lambda_1} = K - q_1 = 0 \rightarrow K = q_1 \\ \frac{\partial \mathcal{L}}{\partial \lambda_2} = K - q_2 = 0 \rightarrow K = q_2 \end{array}\right\} K = q_1 = q_2$

it's just sufficient to solve this system:

$\triangleright\ 400 - 22q_1 - \lambda_1 = 0 \rightarrow 400 - 22q_1 = \lambda_1$

$380 - 22q_1 - (10 - \lambda_1) = 0 \rightarrow 380 - 22q_1 - 10 + 400 - 22q_1 = 0$

$\frac{770}{44} = \frac{44}{44}q_1 = 17{,}5$

$\lambda_1^* = 400 - 22 \cdot 17{,}5 = 15$

Since $K = q_1 = q_2$, we have $K^* = q_1^* = q_2^* = 17{,}5$

$\lambda_2^* = 10 - 15 = -5$

So we started by assuming that $\lambda_1 > 0$, $\lambda_2 > 0$

but we ended up by having a negative $\lambda_2^*$. So

$\lambda_1 > 0$, $\lambda_2 > 0$ it cannot be a valid solution.

For compeleteness we are supposed to also rigourously show elso

- $(\lambda_1 = 0, \lambda_2 > 0)$ and $(\lambda_1 = 0, \lambda_2 = 0)$.

The first case we ruled out since demand $P_1$ is greater than $P_2$, so it's pretty hard to believe that a capacity constraint is affecting the lower demand.

For $(\lambda_1 = 0, \lambda_2 = 0)$ you can easily check that

$$\frac{\partial \mathcal{L}}{\partial q_1} = 400 - 2q_1 - 20q_1 - \lambda_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = 380 - 2q_2 - 20q_2 - \lambda_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial K} = -10 + \lambda_1 + \lambda_2 = 0 \rightarrow -10 = 0$$ does not make sense!

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = K - q_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = K - q_2 = 0$$

So short summary:

given $\lambda_1 \geq 0, \lambda_2 \geq 0$ we need to find the optimal $q_1^*, q_2^*, K^*, \lambda_1^*, \lambda_2^*$

In principle you need to consider all possible combinations $(\lambda_1 = 0, \lambda_2 = 0)$, $(\lambda_1 > 0, \lambda_2 = 0)$, $(\lambda_1 = 0, \lambda_2 > 0)$, $(\lambda_1 > 0, \lambda_2 > 0)$.

· As long as you have that one demand is higher than the other you can easily justify that if a constraint is going to be binding would be in the market with a higher demand. The case ($\lambda_1=0, \lambda_2=0$) it's really easy to check if it's not verified. So we are left with 2 cases that are relevant.

- capacity constraint is binding in the market with higher demand ($\lambda_1>0, \lambda_2=0$)

  Or capacity constraint is binding in both markets ($\lambda_1>0, \lambda_2>0$) $\rightarrow$ we showed that this was not the case

So in our example we have that optimal values are

$\lambda_1^* = 10$, $q_1^* = 17{,}27$, $q_2^* = 17{,}72$, $K^* = 17{,}72$.
$\lambda_2^* = 0$.

How to interpret this results?
if you notice $\lambda_1^* = 10$ which is equal to capacity cost.
$\lambda_1^*$ is the share of the capacity cost c that market 1 pays. In this case the capacity cost is entirely shifted to market 1.
Now let's see if $c = 30$

Lagrangian is $\mathcal{L}(q_1, q_2, K, \lambda_1, \lambda_2) =$

$$(400 - q_1) \cdot q_1 + (380 - q_2) \cdot q_2 - 20(q_1 + q_2) - 30K + \lambda_1(K - q_1) +$$

$$+ \lambda_2 (K - q_2)$$

Kuhn Tucker conditions are

$$\frac{\partial \mathcal{L}}{\partial q_1} = 400 - 2q_1 - 20 q_1 - \lambda_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = 380 - 2q_2 - 20q_2 - \lambda_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial K} = -30 + \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = K - q_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = K - q_2 = 0$$

Remember: $\lambda_1 \geq 0$, $\lambda_2 \geq 0$

Assume $q_1 > 0$, $q_2 > 0$, $K > 0$

Assume $\lambda_2 = 0$

$$\frac{\partial \mathcal{L}}{\partial q_1} = 400 - 2q_1 - 20q_1 - \lambda_1 = 0 \rightarrow \frac{400-30}{22} = \frac{22}{22} q_1 \cong 16,81$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = 380 - 2q_2 - 20q_2 - \lambda_2 = 0 \rightarrow \frac{380}{22} = \frac{\cancel{22}}{\cancel{22}} q_2 \cong 17,27$$

$$\frac{\partial \mathcal{L}}{\partial K} = -30 + \lambda_1 + \lambda_2 = 0 \rightarrow \lambda_1 = 30$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = K - q_1 = 0 \qquad K = q_1 = 16,81$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = K - q_2 = 0$$

By assuming $\lambda_2 = 0$ we have that $\lambda_1 > 0$, which implies that $K^* = q_1^* = 16,81$. So (4) is satisfied. But we have now that $K^* < q_2^*$, since $16,81 < 17,27$, that implies (5) is not satisfied.

Let's try now, $(\lambda_1 > 0, \lambda_2 > 0)$

Now let's consider $\lambda_2 > 0, \lambda_1 > 0$ (4) (5) are binding

$$\frac{\partial \mathcal{L}}{\partial q_1} = 400 - 2q_1 - 20q_1 - \lambda_1 = 0 \rightarrow 400 - 22q_1 - \lambda_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = 380 - 2q_2 - 20q_2 - \lambda_2 = 0 \rightarrow 380 - 22q_1 - (30 - \lambda_1) = 0$$

$$\frac{\partial \mathcal{L}}{\partial K} = -30 + \lambda_1 + \lambda_2 = 0 \rightarrow \lambda_2 = 30 - \lambda_1$$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda_1} &= K - q_1 = 0 \rightarrow K = q_1 \\ \frac{\partial \mathcal{L}}{\partial \lambda_2} &= K - q_2 = 0 \rightarrow K = q_2 \end{aligned}\right\} K = q_1 = q_2$$

it's just sufficient to solve this system:

$\triangleright$ $400 - 22q_1 - \lambda_1 = 0 \rightarrow 400 - 22q_1 = \lambda_1$

$380 - 22q_1 - (30 - \lambda_1) = 0 \rightarrow 380 - 22q_1 - 30 + 400 - 22q_1 = 0$

$\frac{750}{44} = \frac{44}{44} q_1 = 17{,}04$

$\lambda_1 = 400 - 22 \cdot 17{,}04 = 25{,}12$

Since $K = q_1 = q_2$, we have $K = q_1 = q_2 = 17{,}04$

$\lambda_2 = 30 - 25{,}12 = 4{,}88$

So we have $\lambda_2^* = 4{,}88$, $\lambda_1^* = 25{,}12$. $K^* = q_1^* = q_2^* = 17{,}04$.

Let's see if satisfy assumptions and constraints

$\lambda_1^* > 0, \lambda_2^* > 0$ ✓ Assumption is satisfied.

$K^* - q_1^* = 0$ ✓ Constraints are satisfied

$K^* - q_2^* = 0$ ✓

$\lambda_2^* = 4{,}88$, $\lambda_1^* = 25{,}12$.

As yow can see the total cost of capacity in this case with $c=30$, is paid by both markets!